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MATHEMATICAL MODELLING

MSC 34C26

MATHEMATICAL MODELING OF NONLINEAR HEREDITARY OSCILLATORS ON THE EXAMPLE OF DUFFING OSCILLATOR WITH FRACTIONAL DERIVATIVES IN THE SENSE OF RIEMANN-LIOUVILLE

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The paper presents a mathematical hereditary model of Duffing oscillator with friction, which is a generalization of the previously known classical model of Duffing oscillator. This generalization is the replacement of integer derivative by fractional order derivatives in the model equation in the sense of Riemann-Liouville. An explicit finite difference scheme was built to calculate the approximate solution, as well as phase trajectories for different values of control parameters.

Key words: Riemann-Liouville derivative, Grunwald-Letnikov derivative, hereditarity, Duffing oscillator, phase trajectory

Introduction

Investigation of hereditary oscillating systems is one of the topical areas of study which is confirmed by different applications [2]-[5]. Hereditary oscillating systems are considered within the theory of hereditary dynamics suggested by Vito Volterra [2].

Heredity of a process is a property of the process to save the «memory » on its states in previous time points. In particular such processes occur in fractal environments having scale invariance and time and space nonlocality. For the processes in fractal environments, the memory function is exponential, thus such hereditary processes may be described by fractional derivatives. For example, the questions of investigation of hereditary oscillatory systems are described in detail by Riemann-Liouville fractional derivative in the book by Ivan Petras [1].

In the paper we investigate an example of hereditary oscillatory system, hereditary Duffing oscillator with friction and periodic external force. Then, for approximate solution of the model equation with initial conditions, we construct an explicit finite difference scheme. On the basis

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of numerical solution, we build and study oscillograms and phase trajectories of heredity Duffing oscillator.

We should note that a model of Duffing oscillator with fractal friction was suggested in the paper [3].

Statement of the problem

We consider the following hereditary Duffing equation:

$$\frac{d^2}{dt^2}\int_0^t K_1(t-\tau)x(\tau)dt+\alpha\frac{d}{dt}\int_0^t K_2(t-\tau)x(\tau)dt-x(t)+x^3(t)=\sigma\cos(\omega t), \tag{1}$$

where $K_1(t-\tau)$ и $K_2(t-\tau)$ are the memory functions.

If we choose the memory functions in equation (1) in the form:

$$K_1(t-\tau)=\frac{(t-\tau)^{1-p}}{\Gamma(2-p)},\ K_2(t-\tau)=\frac{(t-\tau)^{-q}}{\Gamma(1-q)}, 1<p<2, 0<q<1, \tag{2}$$

we come to the following problem.

Task. *Find a solution $x(t)$, where $t\in[0,T]$ of the following Cauchy problem in local setting [6]:*

$$D_{0t}^{p}x(\tau)+\alpha D_{0t}^{q}x(\tau)-x(t)+x^3(t)=\sigma\cos(\omega t), \tag{3}$$

$$\lim_{t\to 0}t^{2-p}x(t)=x_0, \lim_{t\to 0}\frac{d}{dt}\left(t^{2-p}x(t)\right)=y_0,$$

where $D_{0t}^{q}x(\tau)=\frac{1}{\Gamma(1-q)}\frac{d^2}{dt^2}\int_0^t\frac{x(\tau)d\tau}{(t-\tau)^q}$ *and* $D_{0t}^{p}x(\tau)=\frac{1}{\Gamma(2-p)}\frac{d}{dt}\int_0^t\frac{x(\tau)d\tau}{(t-\tau)^{p-1}}$ *Riemann-Liouville derivatives of fractional orders p, q, α is the viscous fraction coefficient, σ and ω are the amplitude and frequency of periodic external force, x_0 and y_0 are defined constants, initial conditions.*

Owing to the cubic nonlinearity, the Cauchy problem (3) does not have exact solution, so we shall find an approximate solution by the theory of finite difference schemes [6]-[9]. We divide a segment $[0,T]$ into N equal parts with step h. The solution of the differential problem $x(t)$ will turn into an approximate mesh solution $x(t_k)$, $t_k=kh$, $k=1,...,N$. The fractional order derivative in system (3) of discrete Grunvald-Letnikov derivative [10]:

$$D_{0t}^{p}x(\tau)\approx\frac{1}{h^p}\sum_{j=0}^{k-1}m_j^{(p)}x_{k-j}=\frac{x_k}{h^p}+\sum_{j=1}^{k-1}m_j^{(p)}x_{k-j}, \tag{4}$$

$$D_{0t}^{q}x(\tau)\approx\frac{1}{h^q}\sum_{j=0}^{k-1}c_j^{(q)}x_{k-j}=\frac{x_k}{h^q}+\sum_{j=1}^{k-1}c_j^{(q)}x_{k-j},$$

$$c_0^{(q)}=m_0^{(p)}=1, c_j^{(q)}=\left(1-\frac{1+q}{j}\right)c_{j-1}^{(q)}, m_j^{(p)}=\left(1-\frac{1+p}{j}\right)m_{j-1}^{(p)},$$

and integer derivatives

$$\dot{x}(t)=y(t)\approx\frac{x_k-x_{k-1}}{h} \tag{5}$$

Substituting (4) and (5) into system (3), we come to the following numerical solution of the Cauchy problem:

$$\begin{cases} x_k=\frac{1}{B}\left(x_{k-1}-x_{k-1}^3\right)-C\sum\limits_{j=1}^{k-1}m_j^{(p)}x_{k-j}-K\sum\limits_{j=1}^{k-1}c_j^{(q)}x_{k-j}+A\cos(\omega(k-1)h), \\ y_k=\frac{x_k-x_{k-1}}{h}. \end{cases} \tag{6}$$

where $B = h^{-p} + \alpha h^{-q}$, $C = \dfrac{h^{-p}}{B}$, $K = \dfrac{h^{-q}}{B}, A = \dfrac{\sigma}{B}$.

On the basis of paper [10] we may note that approximation (6) of the differential problem (3) is of the first order. In the paper we are not interested in the questions on stability and convergence of the explicit scheme (6). We should only note that explicit schemes are conditionally stable as a rule, i.e. there is a limit for step h. We can estimate the step h by Runge rule [11].

We can also make an experiment to investigate the stability on the right part and the initial data for the chosen control parameter. If the scheme is stable with the first order, it converges with the same order based on Lax theorem. We shall consider some results of the modeling of hereditary Duffing oscillator with friction and harmonic external action.

Results of the modeling

We consider some examples.

Example 1. Control parameter values have the following form: $N = 2000, \sigma = 10, \alpha = 0.15, p = 1.7, q = 0.8, \omega = 5, h = 0.05$.

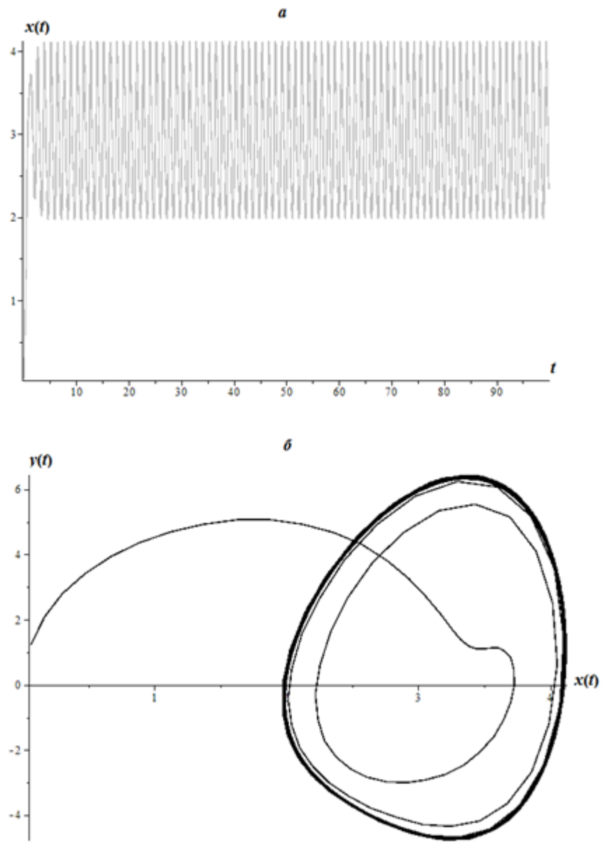

Fig. 1. The curve calculated by formula (6) a) and phase trajectory b)

Fig. 1a shows the calculated curve of the numerical solution obtained by formula (6) and phase trajectories (Fig. 1b). It is clear that the oscillation amplitude almost does not change which may indicate the presence of a periodic solution or a boundary cycle. Indeed, in Fig. 1b phase trajectory reaches the boundary cycle.

Example 2. Control parameters have the following values: $N = 2000, \sigma = 30, \alpha = 0.15, p = 1.7, q = 0.8, \omega = 5, h = 0.05$..

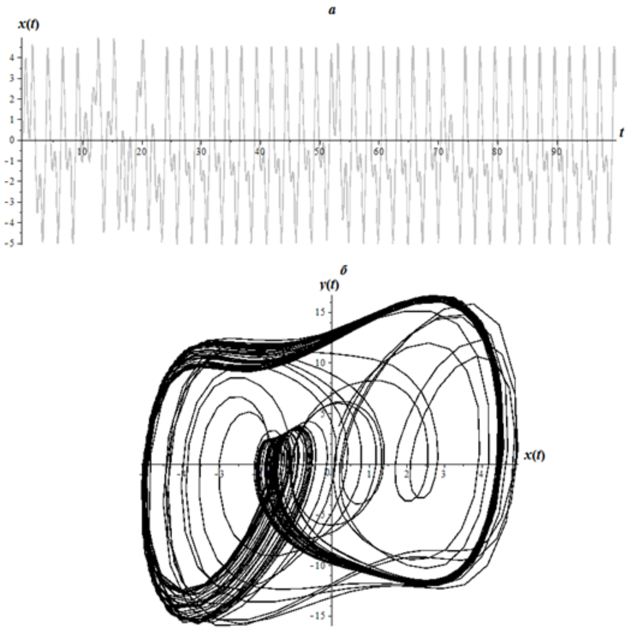

Fig. 2. Curve calculated by formula (6) a) and phase trajectory b)

Fig. 2a shows an oscillogram and a phase trajectory in Fig. 2b, when in comparison to the previous case the amplitude of external force is three times as much. We can see on the oscillogram (Fig. 2a) that oscillations are in chaotic mode with split amplitude at the beginning. Then they reach the quasiregular mode. In Fig. 2b the phase trajectory has a loop corresponding to the split of amplitude oscillations and reaches the boundary cycle.

We consider another example decreasing the sampling interval.

Example 3. Parameters: $N = 2000, \sigma = 30, \alpha = 0.15, p = 1.7, q = 0.8, \omega = 5, h = 0.07$.

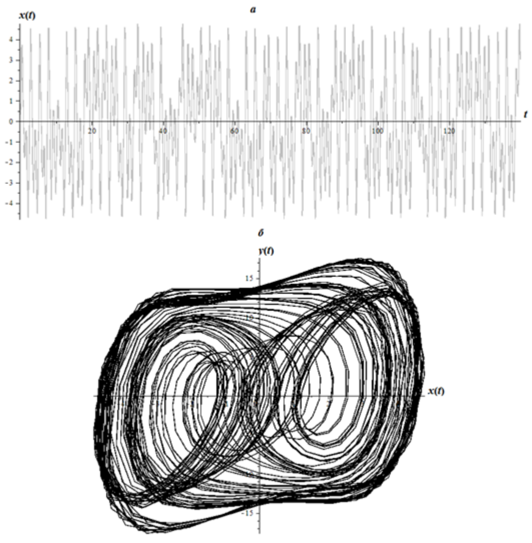

Fig. 3. The curve calculated by formula (6) a) and phase trajectory b)

In this case we see that oscillations occur in a regular chaotic regime. Phase trajectories of the type (Fig. 3b) were obtained in the paper [12] with fractional derivative in the sense of Gerasimov-Caputo.

We consider a case when fractional parameters change.

Example 4. Parameters: $N = 2000, \sigma = 30, \alpha = 0.15, p = 1.3, q = 0.8, \omega = 1, h = 0.07$.

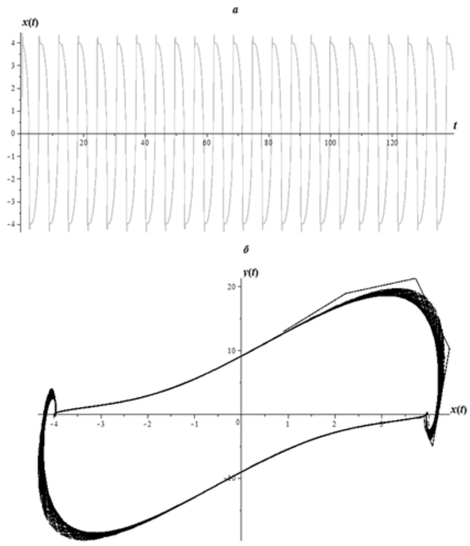

Fig. 4. Curve calculated by formula (6) a) and phase trajectory b)

As parameter p decreases to 1.7, the character of oscillations changes. In Fig. 4a we can notice that the oscillations have split amplitude which is indicated by two loops on the phase trajectory (Fig. 4b), which reaches the boundary cycle. Such a mode of oscillations is typical for Van-der-Pol hereditary oscillator. [12].

These loops are more clearly expressed in the following example.

Example 5. Parameters: $N = 3000, \sigma = 30, \alpha = 0.15, p = 1.6, q = 0.8, \omega = 1, h = 0.07$.

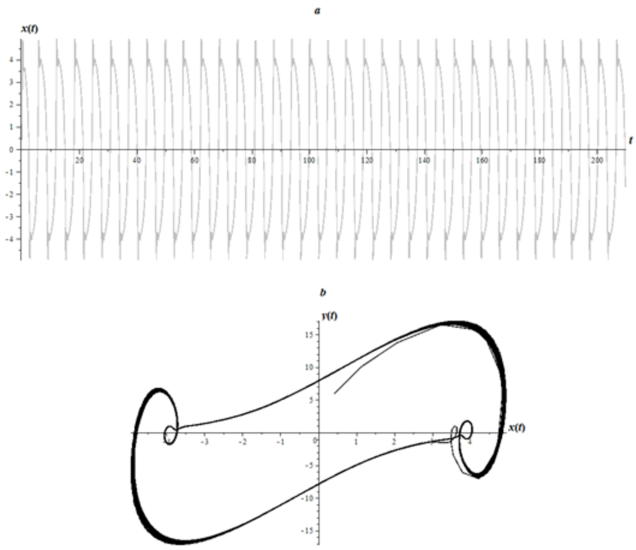

Fig. 5. The curve calculated by formula (6) a) and phase trajectory b)

Amplitude tripartition is observed in the following example.

Example 6. Examples: $N = 2000, \sigma = 30, \alpha = 0.15, p = 1.9, q = 0.8, \omega = 1, h = 0.05$.

In this case we see tripartition of oscillation amplitude (Fig. 6a) that results in additional loops on the phase trajectory (Fig. 6b)

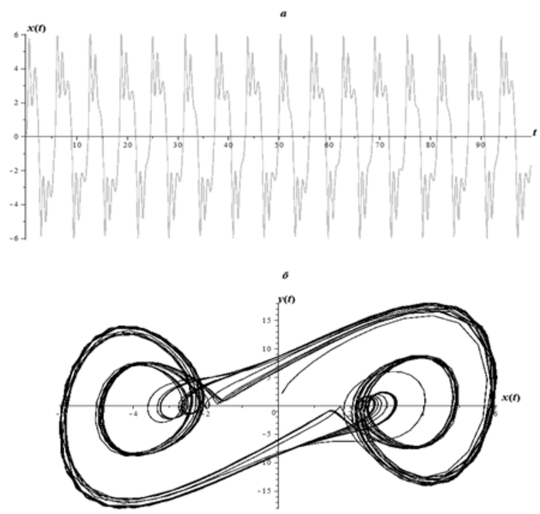

Fig. 6. Curve calculated by formula (6) a) and phase trajectory b)

Conclusions

A mathematical model of hereditary Duffing oscillator with friction and with fractional Riemann-Liouville derivatives was suggested. Explicit finite difference scheme for numerical calculation of an approximated solution of Cauchy problem was constructed in local setting. Taking into account different values of control parameters, oscillograms and phase trajectories were constructed. It is shown that almost all phase trajectories reach the boundary cycle. It was also illustrated that regimes inherent to other oscillating systems may exist. Thus, solution of hereditary Duffing oscillator has wider properties that its classical analogue.

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References

1. Uchaykin V. V. Metod drobnykh proizvodnykh [Method of fractional derivatives] Ul'yanovsk. Artishok. 2008. 512 p.

2. Gao X., Yu J. Chaos in the fractional order periodically forced complex Duffing's oscillators. Chaos, Solitons & Fractals. 2005. vol. 24. no. 4. pp. 1097–1104.

3. Rossikhin Y. A., Shitikova M. V. Application of fractional calculus for dynamic problems of solid mechanics: novel trends and recent results. Applied Mechanics Reviews. 2010. vol. 63. no. 1. 010801.

4. Petras I. Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation. New York. Springer. 2011. 218 p.

5. Syta A., Litak G., Lenci S., Scheffler M. Chaotic vibrations of the Duffing system with fractional damping. Chaos: An Interdisciplinary Journal of Nonlinear Science. 2014. vol. 24. no. 1. 013107.

6. Nakhushev A. M. Drobnoe ischislenie i ego prilozheniya [Fractional calculus and its applications]. Moscow. Fizmatlit. 2003. 272 p.

7. Parovik R. I. Mathematical modeling of nonlocal oscillatory Duffing system with friction. Bulletin KRASEC. Phys. & Math. Sci. 2015. vol. 10. no. 1. pp. 16–21.

8. Parovik R. I. O chislennom reshenii uravneniya fraktal'nogo ostsillyatora s proizvodnoy drobnogo peremennogo poryadka ot vremeni [Numerical solution of fractal oscillatory equation with time derivative

of fractional variable order]. Vestnik KRAUNTs. Fiz.-mat. nauki – Bulletin KRASEC. Phys. & Math. Sci. 2014. no. 1(8). pp. 60–65.

9. Parovik R. I. Numerical analysis of some oscillatory equations with fractional derivative. Bulletin KRASEC. Phys. & Math. Sci. 2014. vol. 9. no. 2. pp. 34–38.

10. Parovik R. I. Ob odnoy konechno-raznostnoy skheme dlya matematicheskoy modeli nelineynogo ereditarnogo ostsillyatora [A finite-difference scheme for a mathematical model of nonlinear hereditary oscillator]. Mezhdunarodnyy nauchno-issledovatel'skiy zhurnal – International Research Journal. 2016. no. 4-2(9). pp. 138–142.

11. Petukhov A. A., Reviznikov D. L. Algoritmy chislennykh resheniy drobno-differentsial'nykh uravneniy [Algorithms of numerical solutions of fractional-differential equations]. Vestnik Moskovskogo aviatsionnogo instituta – Bulletin of the Moscow Aviation Institute. 2009. vol. 16. no. 6. pp. 228–243.

12. Marchuk G.I. Vychislitel'nye metody [Computational Methods]. Moscow. Nauka. 1977. 456 p.

13. Parovik R. I. Ob issledovanii ustoychivosti ereditarnogo ostsillyatora Van-der-Polya [Investigation of the stability of Van-der-Pol hereditary oscillator]. Fundamental'nye issledovaniya – Fundamental Research. 2016. no. 3-2. pp. 283–287.

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