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MATHEMATICAL MODELLING

MSC 35C05

MODELING INVERSIONS WITHIN LOW-MODE MODEL OF LARGE-SCALE DYNAMO

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In this paper we investigate the question of the possibility of inversions within the low-mode dynamo model. The conditions under which the possibility of more frequent reversal of the magnetic field in comparison with inversions in the velocity field of a viscous conducting magnetized fluid.

Key words: low-mode model, the Galerkin method, magnetic inversion

Introduction

The movement of incompressible viscous conductive magnetized fluid is considered in the coordinate system rotating with constant angular velocity $\mathbf{\Omega}$ around Oz axis and placed in external force field with mass density $\mathbf{f}$. The fluid physical parameters are considered to be constant. Magnetohydrodynamic (MHD) equations include Navier-Stokes equation, induction equation for the magnetic field $\mathbf{B}$, continuity equation and solenoidal field $\mathbf{B}$:

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} + 2\Omega\times\mathbf{v} + \Omega\times(\Omega\times\mathbf{r}) = \nu\Delta\mathbf{v} - \dfrac{1}{\rho}\nabla P + \dfrac{1}{\rho\mu}(\nabla\times\mathbf{B})\times\mathbf{B} + \mathbf{f}, \\ \dfrac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \nu_m \triangle \mathbf{B}, \\ \nabla\mathbf{v} = \nabla\mathbf{B} = 0, \end{cases} \tag{1}$$

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where $\mathbf{v}$ is the velocit, P is the pressure, $\mathbf{r}$ is the radius-vector, ρ is the density, ν is the cinematic viscosity, ν_m is the magnetic viscosity, μ is the magnetic permeability.

Since during the constant angular velocity $\mathbf{\Omega}$ the centrifugal force acceleration can be presented in the form $\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = -\frac{1}{2}\nabla(\mathbf{\Omega}\times\mathbf{r})^2$, then it may be joint with the pressure into one potential summand $\frac{1}{\rho}\nabla p'$, where the reduced pressure will be given by the following expression $p' = p + \frac{1}{2}\rho\nabla(\mathbf{\Omega}\times\mathbf{r})^2$.

Let us introduce the following values of velocity U, linear dimensions of region L, time L/U, pressure ρU^2, magnetic induction $L\sqrt{\rho\mu}/U$. Then, in dimensionless variables the system (1) will have the form:

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\,\mathbf{v} = \mathrm{R}^{-1}\Delta\mathbf{v} - \nabla p' - 2\varepsilon^{-1}(\mathbf{e}_z\times\mathbf{v}) + (\nabla\times\mathbf{B})\times\mathbf{B} + \mathbf{f}, \\ \dfrac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \mathrm{R}_m^{-1}\,\triangle\,\mathbf{B}, \\ \nabla\mathbf{v} = \nabla\mathbf{B} = 0, \end{cases} \tag{2}$$

where the dimensionless parameters $R = UL/\nu$ is the Rynolds number, $\varepsilon = U/(\Omega L)$ is the Rossby number, $R_m = UL/\nu_m$ is the Rynolds magnetic number.

The system (2) must be supplemented by boundary conditions for the velocity and magnetic field. The type of boundary conditions is not significant for this paper, so they are not specified. These conditions are assumed to be linear and homogeneous.

One of the approaches to study the system (2) s the construction of Galerkin type few-mode approximations [1]. It should be noted that different versions of dynamo mechanism have the common fact that the source for the field $\mathbf{B}$ toroidal (poloidal) component is the interaction of the poloidal (toroidal) component of the same field with a liquid flow [5, 4]. Thus, it is assumed that the limit possible truncation, maintaining dynamo, is the case with three modes – hydrodynamic, toroidal magnetic, and poloidal magnetic ones.

Let us consider the problem of the choice of a single hydrodynamic mode. If the representation $\mathbf{v}(\mathbf{r},t) = u(t)\mathbf{v}_0(\mathbf{r})$ is used for velocity field and Galerkin approximation for the first equation is derived from the system (2), then the Coriolis approximation term is equal to zero

$$\varepsilon^{-1}u(t)\int(\mathbf{e}_z\times\mathbf{v_0})\,\mathbf{v_0}dV = 0.$$

Thus, information on fluid volume rotation vanishes from the obtained approximation. Owing to this fact, the authors think that the only way to keep the information on the rotation in a velocity single-mode approximation is to use one of the eigen modes of viscous rotating fluid free vibration as the mode. Than the information on the rotation will be included into the streamline form of the considered mode and imaginary component of its eigen value. The real part of this eigen value will determine the time of mode viscous dissipation.

For the chosen nondimensionalization technique such methods are determined from the solution of a spectral problem

$$\begin{aligned} &\Lambda\mathrm{R}^{-1}\mathbf{v} + \mathrm{R}^{-1}\Delta\mathbf{v} - \nabla p' - 2\varepsilon^{-1}(\mathbf{e}_z\times\mathbf{v}) = \mathbf{0}, \\ &\nabla\mathbf{v} = 0. \end{aligned} \tag{3}$$

This spectral problem must be closed by the same boundary conditions for the velocity as the system (2) is.

From the mathematical point of view, the summand $\nabla p'$ in this problem is the projector of the divergence field $2\varepsilon^{-1}(\mathbf{e}_z\times\mathbf{v})$ into solenoidal field space. It is also clear that the problem is a one-parametric one, and is determined by the ratio R/ε – the Coriolis number (Ekman reciprocal number).

As long as the spectrum of the problem (3) s complex, the eigen modes are also complex fields. Thus, in the construction of a few-mode approximation, there is a necessity to consider velocity complex modes and their complex amplitudes. We should also note that if the pare $(\Lambda_0,\mathbf{v}_0)$ is one of the problem (3)solutions, then the pair $\left(\Lambda_0^*,\mathbf{v}_0^*\right)$ will also be the solution for it. Hereafter the asterisk $(*)$ denotes complex conjugation. To present the magnetic field, real modes and their amplitudes are used. Thus, the following presentations of real fields are used in the problem (2):

$$\begin{aligned}\mathbf{v}(\mathbf{r},t)&=u(t)\mathbf{v}_0(\mathbf{r})+u^*(t)\mathbf{v}_0^*(\mathbf{r}),\\ \mathbf{B}(\mathbf{r},t)&=B_1(t)\mathbf{b}_1(\mathbf{r})+B_2(t)\mathbf{b}_2(\mathbf{r}),\end{aligned}\tag{4}$$

where $\mathbf{b}_1(\mathbf{r})$ and $\mathbf{b}_2(\mathbf{r})$ are some toroidal and poloidal magnetic modes, correspondingly.

Let us introduce a scalar product of complex vector fields by the formula

$$\langle\mathbf{p},\mathbf{q}\rangle=\int_D\mathbf{p}\mathbf{q}^*dV,$$

where integration is made by region D volume. We shall further believe that all the considered modes are normalized in terms of the norm of this scalar product.

Galerkin approximation for the system (2), applying the expansion (4), has the form

$$\begin{cases}\dfrac{du}{dt}=C_{11}u^2+C_{12}uu^*+C_{22}u^{*2}-\mathrm{R}^{-1}\Lambda_0u+L_{11}B_1^2+L_{12}B_1B_2+L_{22}B_2^2+F,\\ \dfrac{dB_1}{dt}=W_{11}uB_1+W_{11}^*u^*B_1+W_{12}uB_2+W_{12}^*u^*B_2-\mathrm{R}_m^{-1}\mu_1B_1,\\ \dfrac{dB_2}{dt}=W_{21}uB_1+W_{21}^*u^*B_1+W_{22}uB_2+W_{22}^*u^*B_2-\mathrm{R}_m^{-1}\mu_2B_2,\end{cases}\tag{5}$$

where C_{ij}, L_{ij}, F, W_{ij}, μ_i coefficients are generally complex numbers which are the following scalar products of vector fields:

$$\begin{aligned}C_{11}&=-\langle(\mathbf{v}_0\nabla)\,\mathbf{v}_0,\mathbf{v}_0\rangle,\\ C_{12}&=-\left\langle(\mathbf{v}_0\nabla)\,\mathbf{v}_0^*,\mathbf{v}_0\right\rangle-\left\langle\left(\mathbf{v}_0^*\nabla\right)\mathbf{v}_0,\mathbf{v}_0\right\rangle,\\ C_{22}&=-\left\langle\left(\mathbf{v}_0^*\nabla\right)\mathbf{v}_0^*,\mathbf{v}_0\right\rangle,\\ L_{11}&=\langle(\nabla\times\mathbf{b}_1)\,\mathbf{b}_1,\mathbf{v}_0\rangle,\\ L_{12}&=\langle(\nabla\times\mathbf{b}_1)\,\mathbf{b}_2,\mathbf{v}_0\rangle+\langle(\nabla\times\mathbf{b}_2)\,\mathbf{b}_1,\mathbf{v}_0\rangle,\\ L_{22}&=\langle(\nabla\times\mathbf{b}_2)\,\mathbf{b}_2,\mathbf{v}_0\rangle,\\ F&=\langle\mathbf{f},\mathbf{v}_0\rangle,\\ W_{ij}&=\left\langle\nabla\times\left(\mathbf{v}_0\times\mathbf{b}_j\right),\mathbf{b}_i\right\rangle,\\ \mu_1&=-\langle\Delta\mathbf{b}_T,\mathbf{b}_T\rangle>0,\quad \mu_2=-\langle\Delta\mathbf{b}_P,\mathbf{b}_P\rangle>0,\end{aligned}\tag{6}$$

Developing the system (5) we considered [3], that toroidal and poloidal field spaces are invariant relatively the Laplace operator, toroidal and poloidal field orthogonality as well as Laplace operator positiveness.

The magnetic modes are assumed to ensure mutual generation of each other not generating themselves. Mathematically it means that coefficients $W_{ii} = 0$, and then the equation (5) take on the form:

$$\begin{cases} \dfrac{du}{dt} = C_{11}u^2 + C_{12}uu^* + C_{22}u^{*2} - \mathrm{R}^{-1}\Lambda_0 u + L_{11}B_1^2 + L_{12}B_1B_2 + L_{22}B_2^2 + F, \\ \dfrac{dB_1}{dt} = (W_{12}u + W_{12}^* u^*)B_2 - \mathrm{R}_m^{-1}\mu_1 B_1, \\ \dfrac{dB_2}{dt} = (W_{21}u + W_{21}^* u^*)B_1 - \mathrm{R}_m^{-1}\mu_2 B_2, \end{cases} \tag{7}$$

Investigation of MHD equation system in Galerkin type few-mode approximation for the presence of inversions

Inversion in any field under the consideration is expressed in corresponding mode sign change, and the problem on determination of the conditions and parameters, when magnetic field inversion takes place at the background of the absence of any inversions in the velocity field of a viscous incompressible fluid, is reduced to the selection of such coefficients in the system (7), that the mode $B_1(t)$ and $B_2(t)$ sign would change quite frequently at quite long time intervals of mode$u(t)$ and $u^*(t)$ constant sign.

To simplify the transformations in the course of the study, coefficient notation changes $\Lambda = \mathrm{R}^{-1}\Lambda_0,\ \mu_1 = \mathrm{R}_m^{-1}\mu_1,\ \mu_2 = \mathrm{R}_m^{-1}\mu_2$ are introduced into the system (7) and it is written as follows:

$$\begin{cases} \dfrac{du}{dt} = C_{11}u^2 + C_{12}uu^* + C_{22}u^{*2} - \Lambda u + L_{11}B_1^2 + L_{12}B_1B_2 + L_{22}B_2^2 + F, \\ \dfrac{dB_1}{dt} = (W_{12}u + W_{12}^* u^*)B_2 - \mu_1 B_1, \\ \dfrac{dB_2}{dt} = (W_{21}u + W_{21}^* u^*)B_1 - \mu_2 B_2, \end{cases} \tag{8}$$

Let us investigate the parameters of the obtained system (8), when magnetic field inversion is possible in the condition of relative constancy of velocity field.

Knowing the mode $u(t)$ and $u^*(t)$ vales from the differential equation system formed by the two last equations from (8), we determine the conditions when mode $B_1(t)$ and $B_2(t)$ sign change may occur, i.e. the magnetic field inversion:

$$\begin{cases} \dfrac{dB_1}{dt} = (W_{12}u + W_{12}^* u^*)B_2 - \mu_1 B_1, \\ \dfrac{dB_2}{dt} = (W_{21}u + W_{21}^* u^*)B_1 - \mu_2 B_2. \end{cases} \tag{9}$$

In the system (9) oscillations may appear only in the case when the solution has the following form

$$B_i(t) = \alpha_i e^{k_i t},\ k_i \in \mathbb{C}, \tag{10}$$

where k_i is the complex root of the system (9) characteristic equation:

$$\begin{vmatrix} -\mu_1 - k & W_{12}u + W_{12}^* u^* \\ W_{21}u + W_{21}^* u^* & -\mu_2 - k \end{vmatrix} = 0$$

with negative discriminant

$$D = (\mu_2 - \mu_1)^2 + 16\Re(W_{12}u)\Re(W_{21}u) < 0 \tag{11}$$

where

$$\begin{aligned} u &= \Re(u) + i\Im(u), \\ u^* &= \Re(u) - i\Im(u), \\ W_{12} &= \Re(W_{12}) + i\Im(W_{12}), \\ W_{12}^* &= \Re(W_{12}) - i\Im(W_{12}), \\ W_{21} &= \Re(W_{21}) + i\Im(W_{21}), \\ W_{21}^* &= \Re(W_{21}) - i\Im(W_{21}). \end{aligned}$$

On the basis of the condition (11), the range of mode $u(t)$ values on a complex plane is the range limited by a hyperbolic type line with a symmetry center at the beginning of the coordinates. Generally, for any $u(t)$ value from the given range, mode $B_1(t)$ and $B_2(t)$ values may be randomly chosen that follows from the solution (10). Mode $B_1(t)$ and $B_2(t)$values are defined so that the equality $B(t) = \sqrt{B_1^2(t) + B_2^2(t)} = 1$ is satisfied which will allow us to study the mutual change of mode $B_1(t)$ and $B_2(t)$ values as well as the change of amplitude $B(t)$ value.

If the dissipative terms (Λu, $\mu_1 B_1$, $\mu_2 B_2$) and external influence (F) are not taken into the account in the system (8), the process of field mutual generation will be continuous(Fig. 1). Elimination only of external influence from the system (8) depending on parameter values of the equation will result in the process attenuation (Fig.2) or continuous process (Fig.3).

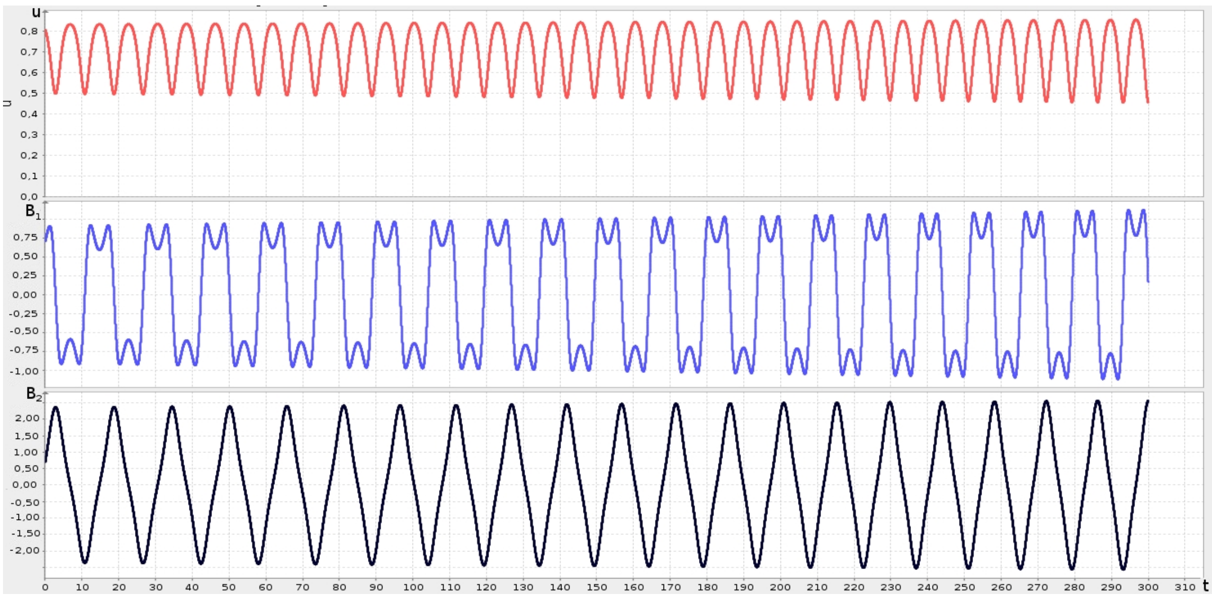

Fig. 1. Visualization of system (8) computational solution without dissipative terms and external influence ($\lambda = 10^{-3}$, $\mu = 4$).

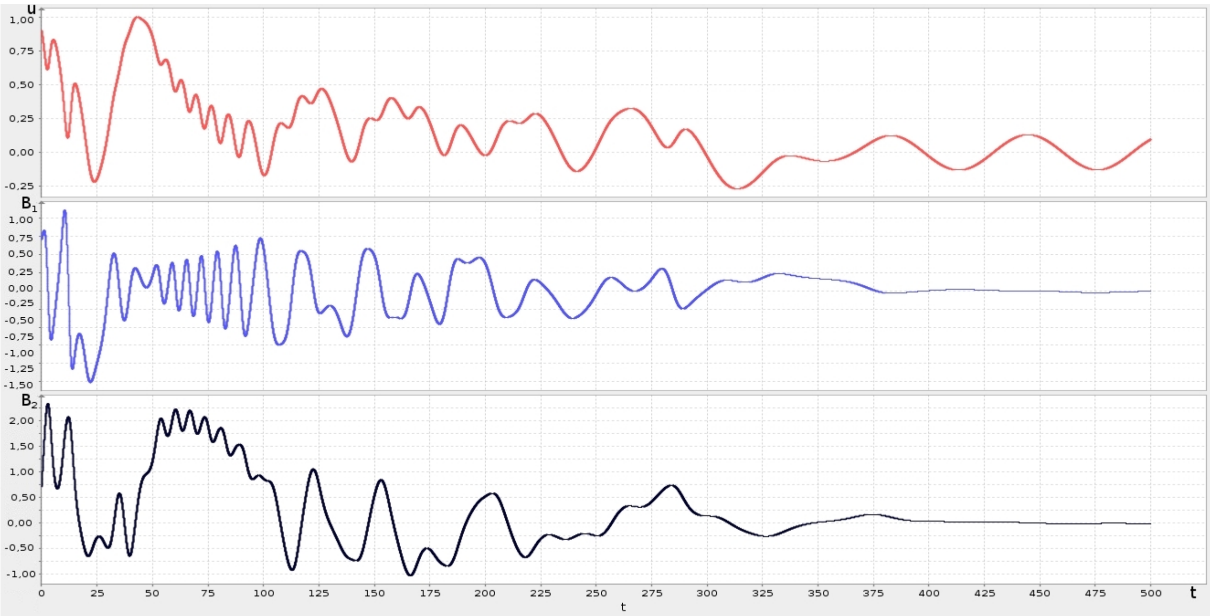

Fig. 2. Visualization of system (8) computational solution without external influence ($\lambda = 10^{-6}$, $\mu = 10^{-2}$).

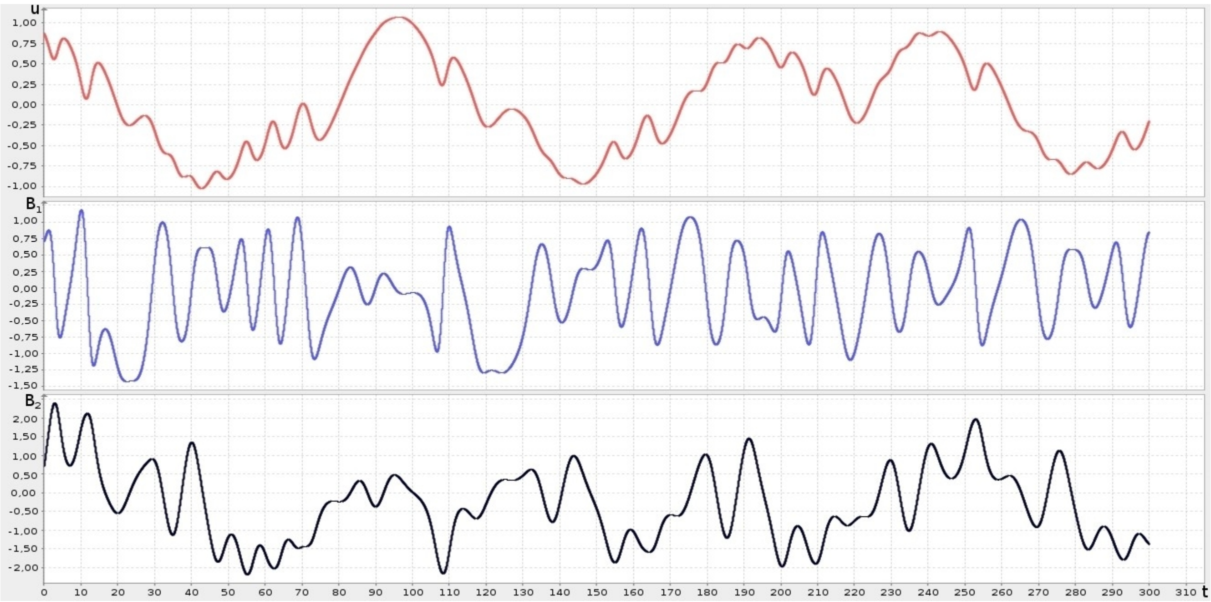

Fig. 3. Visualization of system (8) computational solution without external influence ($\lambda = 10^{-7}$, $\mu = 10^{-3}$).

Conclusions

Galerkin type MGD equation system in the few-mode approximation (8), where the external influence is not taken into the account for different parameter values, contains computational solutions with inversions both in magnetic field and in velocity field of a viscous fluid. There are cases of field attenuation and continuous mutual generation which indicates the adequacy of the description of the Earth core processes within the few-mode model.

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